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A comparative study of interacting random-walk models

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Abstract. We analyse four interacting random-walk models from a comparative point of view, concentrating on one-dimensional versions. We consider an interacting random-walk model recently introduced by Stanley *et al*, the Domb–Joyce model, and the 'true' self-avoiding walk of Amit *et al*. In addition, we introduce a model based on the weighting of turning points, whose properties are related to those of the Ising spin chain. Fixed points and their stability are identified for both attractive and repulsive correlations in all four cases. We compare the mechanisms involved in each of these models and comment on the role played in each by its particular form of correlation between steps. The different universal behaviours are shown to arise from competition between short-range and long-range (and particularly cumulative) memory effects, and between local and global normalisation conditions.

In the past few years, a considerable amount of work has been done on a number of 'interacting random-walk' models. These differ from the well known non-interacting random walk model (applicable for example to brownian motion) in that they incorporate suitably chosen correlations between steps. This is done in order to account for particular features of physical systems or processes, whose complexity goes beyond what can be obtained from a simple brownian-motion picture. An early example is that of the 'self-avoiding walk' (SAW); in its simplest version, it is a random walk subject to the condition that a given site cannot be visited more than once. This constraint simulates the excluded volume repulsion acting between segments of a flexible polymer chain; indeed the SAW turns out to provide an accurate model for the configurations of isolated polymer chains in a good solvent, this being a system where excluded volume is the only relevant parameter (see e.g. de Gennes 1979 and references therein).

The Domb–Joyce model for polymer chains (Domb and Joyce 1972) was an attempt to interpolate between the random walk and the SAW. In it, each self-intersection has a weight $1-w$ between 0 and 1 (more precise definitions are given below). It has been found that, for any $w>0$, the asymptotic behaviour of the Domb–Joyce model is the same as that of the SAW (which corresponds to $w=1$); see Domb (1983), and references therein. At $w=0$ one recovers the uncorrelated random walk.

More recently, two variants of interacting random walks have received special attention: one is a model introduced by Stanley *et al* (1983), in which each new site

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visited has a weight p and which exhibits certain hyperuniversal, that is, dimension-independent, properties, much the same as in the problem of diffusion in randomly porous media (see e.g. Alexander and Orbach 1982). The other is the 'true' self-avoiding walk (TSAW) defined by Amit *et al* (1983) as the problem of the traveller who steps at random but tries to avoid places he has already visited; it has been proposed that the two-dimensional TSAW could be a model for polymer adsorption on a surface under suitable conditions (Bulgadaev and Obukhov 1983).

Besides their potential applicability to concrete physical situations, these models display some peculiar features, such as the fact that they belong to different universality classes from the random walk or the ordinary SAW (although they do cross over either to random walk or SAW behaviour in suitable limits). The question can then be raised as to whether one can build a unified picture of the mechanisms underlying the various crossovers in the above mentioned models. Such a picture would be of help towards a better theoretical understanding of generalised random-walk problems. In what follows, we analyse four interacting random walk models from a comparative point of view, discussing the role played in each by its particular form of correlation between steps. We shall concentrate on one-dimensional versions; this is because, while already exhibiting non-trivial behaviour, these are usually amenable to exact (or at least asymptotic) solutions; even for models in which such solutions do not exist, their one-dimensional character makes it easy to obtain definite evidence about qualitative behaviour from e.g. series studies (Stella *et al* 1984; see also below).

We first recall the main features of both the interacting random-walk model of Stanley *et al* (1983) and of the TSAW. We then introduce a one-dimensional short-range interacting model, based on weighting a random walk according to how many times the walker changes direction (we call it the 'turning-point' model), and explore its relationship to other models. The one-dimensional Domb–Joyce model is then discussed (for the first time, as far as our knowledge goes) in both the repulsive and attractive limits. Finally, a discussion of the mechanisms involved in each of these models is given, and general comments are made about the behaviour of interacting random walk models.

The model introduced by Stanley *et al* (1983) is a random walk, where to each new site visited a weight p is attributed, so a walk which visits s distinct sites has a weight p^s. The correlation between steps is thus attractive or repulsive depending on whether p is smaller or greater than one; at $p=1$ the uncorrelated random walk is recovered. In the repulsive case one finds that, if the number N of steps is sufficiently large, the dominant behaviour is that of an ordinary SAW (Stanley *et al* 1983). This means that the average end-to-end distance is $\langle R_N^2\rangle^{1/2}\sim N^{\nu}$, where $\nu\cong 3/(d+2)$ for space dimensionality $1<d\leq 4$ (equality holds at $d=1$ and 4), and that for $d>d_c=4$, the upper critical dimensionality of SAWs, repulsion becomes irrelevant and the pure random-walk behaviour dominates with $\nu=\frac{1}{2}$ (de Gennes 1979). For attractive correlation, Stanley *et al* (1983) point out that the mean-square displacement appears to saturate at a finite value in two and three dimensions; in one dimension, the saturation effect does not occur. Actually, the one-dimensional case has been asymptotically solved (Redner and Kang 1983), and it has been found that for $0<p<1$ the average number of visited sites $\langle S_N\rangle$ is proportional to $N^{1/3}$; it is expected that in one dimension, the root-mean-square displacement should scale as $\langle S_N\rangle$, so $\langle R_N^2\rangle^{1/2}\sim N^{1/3}$ in the attractively correlated one-dimensional walk.

The 'true' self-avoiding walk (TSAW) is defined as follows (Amit *et al* 1983): on a lattice, the traveller has to move to one of the Z first neighbours of the site he is at.

The probability P_i of moving to a site i depends on the number of times n_i this site has already been visited:

$$P_i = \mathrm{e}^{-gn_i} \Big/ \sum_{j=1}^{z} \mathrm{e}^{-gn_j}. \qquad (1)$$

The parameter $g > 0$ defines the strength with which the walk repels itself. One can also make an extension to the attractive case $g < 0$; for $g = 0$ one recovers the ordinary random walk.

The upper critical dimensionality of the TSAW is two (Amit *et al* 1983); from a self-consistent approach, Pietronero (1983) obtained an explicit approximate expression for the end-to-end distance exponent ν, for $d \leq 2$ in the repulsive case, namely:

$$\nu = 2/(d+2) \qquad d \leq 2. \qquad (2)$$

He does not consider attractive correlations. Equation (2) gives $\nu = \frac{2}{3}$ in one dimension, a result which is expected to hold as long as the repulsion parameter g is finite; in the infinite repulsion limit, one has the ordinary one-dimensional SAW, with $\nu = 1$ (Pietronero 1983). This view is supported by the scaling and crossover arguments of de Queiroz *et al* (1984). In dimensions greater than one, there is no change of universality class at $g = \infty$ (Amit *et al* 1983, de Queiroz *et al* 1984, Stella *et al* 1984; see also the Monte Carlo simulations of Angles d'Auriac and Rammal (1984) on a Sierpinski gasket). The value of $\nu = \frac{2}{3}$ for positive, finite g in one dimension has been given support by Monte Carlo simulations (Bernasconi and Pietronero 1983, Rammal *et al* 1984), exact enumeration (Stella *et al* 1984) and scaling arguments (Obukhov 1984). It has been found that in one dimension the attractive TSAW ($g < 0$) is always self-trapping (Stella *et al* 1984, Rammal *et al* 1984).

The above picture indicates that the effects of correlation between steps in the one-dimensional TSAW are more drastic in the attractive case than in the repulsive one: the smallest degree of attraction causes the TSAW to trap itself, whereas one needs infinite repulsion to recover the ordinary SAW behaviour. On the other hand, the interacting random walk of Stanley *et al* (1983) displays features that are in a sense complementary to those of the TSAW: in one dimension, it will only trap itself for infinite attraction, and it will behave as an ordinary SAW for any non-zero repulsion.

At this point, it is important to emphasise a basic difference between the TSAW and the ordinary SAW: in the former, it is a *local* criterion provided by equation (1) above which tends to drive the walker away from his previous steps, whereas for the latter, it is the *total* number of intersections that matters (Amit *et al* 1983, Pietronero 1983). Indeed, when used in the context of an effective-medium approximation these criteria have led respectively to the expression $\nu = 2/(d+2)$, $d \leq 2$, for the TSAW and to the well known Flory value $\nu = 3/(d+2)$, $d \leq 4$, for the ordinary SAW (Pietronero 1983). A moment's thought shows that it is also the total number of self intersections that plays the important role in the model of Stanley *et al* (1983); hence it is not surprising that in the repulsive regime it behaves as an ordinary SAW. (That is, $\langle R_N^2 \rangle^{1/2} \sim N^{\nu}$ with $\nu = 1$ in $d = 1$ for large N.) This is also the case in the Domb–Joyce model (Domb 1983; see below).

The question then arises of why the TSAW differs from a pure non-interacting random walk. We argue that this effect is due to an *infinitely long-range* memory in addition to the local criterion described above. In order to support this argument, we now proceed to discuss an interacting random walk model in which a local criterion is used, but which displays a *short-range* memory effect, and see how it differs from the TSAW.

Consider a one-dimensional random walk in which each step has a different probability according to whether it is in the same direction as or opposite to the immediately preceding one. This problem bears a formal analogy to the Ising spin chain: if $\sigma_i = \pm 1$ denotes a step in the positive or negative direction, the correlation described above can be written as:

$$P(\sigma_i) = e^{g\sigma_i\sigma_{i-1}}/(e^g + e^{-g}) \tag{3}$$

where $P(\sigma_i)$ is the probability that the ith step be given in the σ_i direction, and depends on the direction of the previous step σ_{i-1}; the parameter g characterises the strength of the correlation. Again $g = 0$ corresponds to the unbiased random walk, and $g = \pm\infty$ correspond respectively to a 'ferromagnetic' or 'antiferromagnetic' ground state. Since the intrinsic probability of an N-step configuration is given by the product of the probabilities of each of its steps, it can easily be seen that the average squared end-to-end distance is

$$\langle R_N^2 \rangle = \frac{1}{Z_N} \operatorname*{Tr}_{\{\sigma_i\}} \left(\sum_{i,j=1}^{N} \sigma_i \sigma_j \right) \exp\left[g \sum_{i=2}^{N} \sigma_i \sigma_{i-1} \right] \tag{4}$$

with

$$Z_N = \operatorname*{Tr}_{\{\sigma_i\}} \exp\left[g \sum_{i=2}^{N} \sigma_i \sigma_{i-1} \right]. \tag{5}$$

That is to say, the end-to-end distance in this model is related to a correlation function of the Ising chain; this can be evaluated using standard transfer-matrix methods. If periodic boundary conditions are assumed (the precise boundary condition turns out to be of no importance here), one has:

$$\langle R_N^2 \rangle = N\, e^{2g}(1 - t^N)/(1 + t^N) \tag{6}$$

where $t \equiv \tanh g$. For any finite g (positive, zero or negative) and $N \to \infty$, $\langle R_N^2 \rangle = N\, e^{2g}$, whence the behaviour is always that of a standard random walk, with the correlation-length exponent $\nu = \frac{1}{2}$; the amplitude e^g gives the size of an effective step in the scaled uncorrelated random walk. From (6) one also obtains $\langle R_N^2 \rangle = N^2$ as $g \to +\infty$ and $\langle R_N^2 \rangle \to 0$ or $\langle R_N^2 \rangle = 1$ for $g \to -\infty$, depending on whether N is even or odd (more precisely, these limits are obtained if $e^{2|g|} \gg N \gg 1$, that is, the crossover variable is $N\, e^{-2|g|}$).

We have thus seen that it is the combination of a local criterion for the weighting of steps plus an infinite-range memory effect which in the repulsive regime casts the TSAW between the SAW and the random walk. On the other hand, the random walk behaviour of the turning point model is due to a local criterion plus finite-range memory.

In the Domb–Joyce model, it is a global criterion that determines the weight of a walk, namely, if $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ are the positions of lattice sites occupied by the ith and jth steps of the walk, an N-step configuration has the weight;

$$\prod_{i=0}^{N-2} \prod_{j=i+2}^{N} (1 - w\delta(\boldsymbol{r}_i, \boldsymbol{r}_j)). \tag{7}$$

It can be seen from (7) that in this case, the memory effect is not only infinitely long-ranged, but also cumulative: if a site is visited n times, it contributes a factor $(1-w)^{n(n-1)/2}$ to the weight of the walk; a cumulative behaviour is also present in the

TSAW, as seen from equation (1) above. In the model of Stanley *et al* (1983) this is not the case: there, it does not matter how many times a site has been visited, only whether it has been visited or not.

The Domb–Joyce model has been extensively studied in three dimensions (Domb 1983); to our knowledge, its one-dimensional version has not been discussed, perhaps due to one's intuition that it should be trivial. However, we felt that it had to be checked, in order to widen the frame of the present comparative study. We have done exact enumeration of the average squared end-to-end distance of the one-dimensional Domb–Joyce model, for walks of up to 21 steps, both in the repulsive ($0 < w < 1$) and attractive ($w < 0$) cases. Indeed, in the repulsive limit the behaviour crosses over to that of the SAW, as in higher dimensions. On the other hand, although it traps itself for any $w < 0$, it has a different behaviour from that of the attractive TSAW: there, one has a saturation of both the average number of visited sites S_N, and of the end-to-end distance R_N(Rammal *et al* 1984); in the Domb–Joyce case, the average maximum extent of the walk saturates, but the end-to-end distance seems to collapse. Further work is in progress along these lines, and will be published elsewhere.

In figure 1 we show schematically the universality classes for the Domb–Joyce model, the TSAW, the interacting random walk and the turning-point model, all of them for both repulsive and attractive correlations.

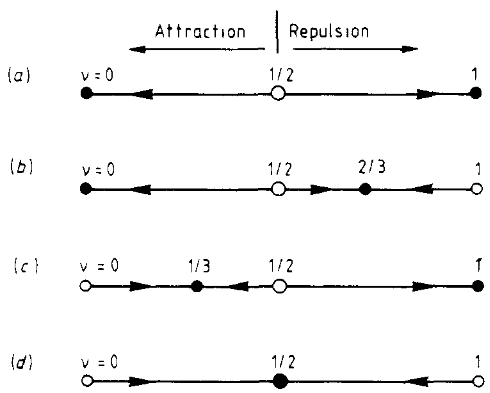

Figure 1. Universality classes for: (*a*) Domb–Joyce model; (*b*) TSAW; (*c*) interacting random walk of Stanley *et al* (1983); (*d*) turning-point model defined in the text. All for one dimension; full (open) circles denote stable (unstable) fixed points.

A comparative analysis of these models can be made in terms of an energy–entropy balance: in general, the least energetic configuration will be either a completely stretched (recall that we are restricting to one-dimensional problems) or a 'piled-up' one, depending on whether correlations are repulsive or attractive. Both have zero entropy; on the other hand, a random-walk configuration (whose entropy is maximised) is likely to be highly energetic. The specific way in which the energy term is written, and the way in which it compares to the entropic term, determine which one dominates, or whether both are of the same order of magnitude (in which case an intermediate behaviour appears).

Thus, for finite g, the 'turning-point' model, as already known from the thermodynamics of the Ising chain, is always in a high-temperature phase; random-walk behaviour dominates everywhere except in the infinite-correlation (zero-temperature) limits.

The introduction of the infinite-range (though not cumulative) effect, together with the weighting of the total number of visited sites, in the model of Stanley *et al* (1983) gives more importance to the energy term than in the turning-point model; in the repulsive case, energy always overwhelms entropy and one has SAW behaviour; in the attractive limit energy and entropy balance to give the intermediate behaviour $\langle S_N \rangle \sim N^{1/3}$ (Redner and Kang 1983). The entropic term has a stronger effect in the attractive case because of the particular shape of the distribution curve for $P_N(s)$ (number of N-step walks that visit s sites) in the limit $s/N \ll 1$ (Redner and Kang 1983).

In contrast, the fact that the memory effect is cumulative in the TSAW makes it saturate in the attractive case. However, the repulsive case with finite repulsion does not cross over to the SAW, because of the peculiar local criterion of weighting steps: provided that the repulsion is finite, the walk will eventually turn back on itself, and equation (1) shows that in one dimension it can keep going backwards indefinitely without further obstacles (de Queiroz *et al* 1984).

Finally, with a global weighting of walks, and a long-range cumulative memory effect, energy always dominates in the Domb–Joyce model and one has either the SAW in the repulsive case or a trapping regime (perhaps even more drastic than for the TSAW, as pointed out above) in the attractive one.

In summary, we have made a comparative study of some correlated random-walk models; from the mechanisms involved, we have seen that local normalisation conditions tend to favour the entropic term, whereas global normalisation seems to favour energy; long-range memory favours energy, and this is enhanced if the memory is cumulative. We hope that this work will be relevant as a first step towards a unified view of various random-walk related models (up to now considered mostly separately).

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